Asymptotic of sequences A161630, A212722, A212917 and A245265

(*Václav Kotěšovec, published July 16 2014*)

In the OEIS (On-Line Encyclopedia of Integer Sequences) published Paul D. Hanna in 2009 sequences A161630, A212722, A212917 and I added in 2014 sequence A245265, which can be generalized (for $p \geq 1$) as the family of the sequences with an exponential generating function A(x), satisfies functional equation

$$A(x) = exp\left(\frac{x}{1 - x\, A(x)^p}\right)$$

or as the sum

$$a_n = \sum_{k=0}^{n} \frac{n!}{k!} * \binom{n-1}{n-k} * \left(1 + p * (n-k)\right)^{k-1}$$

Theorem (V. Kotěšovec, July 15 2014):
Asymptotic

$$a_n \sim \frac{n^{n-1}\, p^{n-1+\frac{1}{p}} \left(1 + 2 * LambertW\left(\frac{\sqrt{p}}{2}\right)\right)^{n+\frac{1}{2}}}{e^n\, 2^{2*n+\frac{2}{p}}\ \ LambertW\left(\frac{\sqrt{p}}{2}\right)^{2*n+\frac{2}{p}-\frac{1}{2}} \sqrt{1 + LambertW\left(\frac{\sqrt{p}}{2}\right)}}$$

Proof:

Following theorem by Edward A. Bender is (in case of implicit functions) very useful (for proof see [1], p.505 and also [4], p.469).

Citation: Edward A. Bender, "Asymptotic methods in enumeration" (1974), p.502, see [1]

THEOREM 5. *Assume that the power series $w(z) = \sum a_n z^n$ with nonnegative coefficients satisfies $F(z, w) \equiv 0$. Suppose there exist real numbers $r > 0$ and $s > a_0$ such that*

(i) *for some $\delta > 0$, $F(z, w)$ is analytic whenever $|z| < r + \delta$ and $|q| < s + \delta$;*
(ii) $F(r, s) = F_w(r, s) = 0$;
(iii) $F_z(r, s) \neq 0$, *and* $F_{ww}(r, s) \neq 0$: *and*
(iv) *if* $|z| \leqq r, |w| \leqq s$, *and* $F(z, w) = F_w(z, w) = 0$, *then* $z = r$ *and* $w = s$.

Then

(7.1) $$a_n \sim ((rF_z)/(2\pi F_{ww}))^{1/2} n^{-3/2} r^{-n},$$

where the partial derivatives F_z and F_{ww} are evaluated at $z = r$, $w = s$.

Bender's formula modified for exponential generating function is

$$\frac{a_n}{n!} \sim \frac{1}{n\, r^n} \sqrt{\frac{r\, F_z}{2\pi n\, F_{ww}}}$$

$$a_n \sim \frac{n^{n-1}}{e^n\, r^{n-1/2}} \sqrt{\frac{F_z}{F_{ww}}}$$

Now we have the implicit function

$$f(x,y) = e^{\frac{x}{1-xy^p}} - y$$

partial derivatives		
F_z	$\frac{\partial}{\partial x} f(x,y)$	$\frac{e^{\frac{x}{1-xy^p}}}{(xy^p-1)^2}$
F_w	$\frac{\partial}{\partial y} f(x,y)$	$\frac{px^2y^{p-1}e^{\frac{x}{1-xy^p}}}{(xy^p-1)^2} - 1$
F_{ww}	$\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)$	$-\frac{px^2y^{p-2}e^{\frac{x}{1-xy^p}}(p(x^2y^p(y^p-1)-1)+(xy^p-1)^2)}{(xy^p-1)^4}$

r, s, are roots of the system of equations

$$e^{\frac{r}{1-rs^p}} - s = 0 \qquad \frac{pr^2s^{p-1}e^{\frac{r}{1-rs^p}}}{(rs^p-1)^2} - 1 = 0$$

From first equation we have

$$r = \frac{1}{s^p + \frac{1}{\log(s)}}$$

and second equation can be reduced

$$pr^2s^p = (rs^p-1)^2$$

$$p\, s^p\, (log(s))^2 = 1$$

$$s = e^{\frac{2\,W\left(\frac{\sqrt{p}}{2}\right)}{p}} = \left(\frac{\sqrt{p}}{2\,W\left(\sqrt{p}/2\right)}\right)^{\frac{2}{p}}$$

where W is the LambertW function

$$r = \frac{4\,\left(W\left(\sqrt{p}/2\right)\right)^2}{p\,\left(1+2\,W\left(\sqrt{p}/2\right)\right)}$$

The asymptotic is then

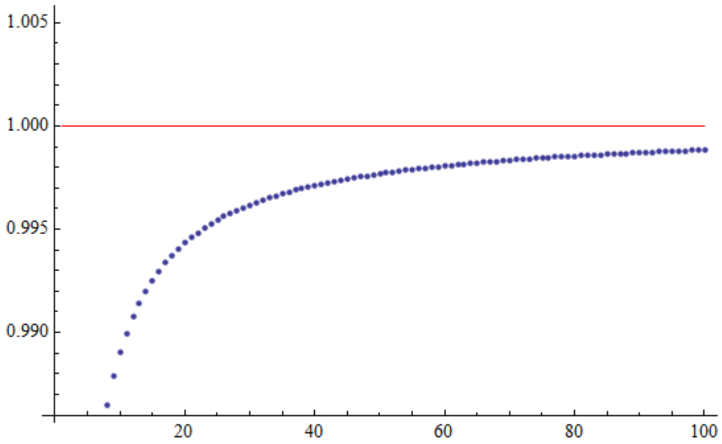
$$a_n \sim \frac{n^{n-1}}{e^n\, r^{n-1/2}}\sqrt{\frac{F_z}{F_{ww}}} = e^{-n}n^{n-1}r^{-n}\sqrt{-\frac{s^{2-p}(rs^p-1)^2}{pr(p(r^2s^p(s^p-1)-1)+(rs^p-1)^2)}}$$

and after simplification

$$a_n \sim \frac{n^{n-1}\, p^{n-1+\frac{1}{p}}\, \left(1+2*W\left(\sqrt{p}/2\right)\right)^{n+\frac{1}{2}}}{e^n\, 2^{2*n+\frac{2}{p}}\, W\left(\sqrt{p}/2\right)^{2*n+\frac{2}{p}-\frac{1}{2}}\sqrt{1+W\left(\sqrt{p}/2\right)}}$$

Numerical verification (for p=4, sequence A245265)

```
Show[
 ListPlot[
  Table[Sum[n! * (1 + p * (n - k)) ^ (k - 1) / k! * Binomial[n - 1, n - k], {k, 0, n}] /
     (p ^ (n - 1 + 1 / p) * (1 + 2 * LambertW[Sqrt[p] / 2]) ^ (n + 1 / 2) *
       n ^ (n - 1) / (Sqrt[1 + LambertW[Sqrt[p] / 2]] * E ^ n * 2 ^ (2 * n + 2 / p) *
          LambertW[Sqrt[p] / 2] ^ (2 * n + 2 / p - 1 / 2))) /. p → 4, {n, 1, 100}]],
 Plot[1, {n, 1, 100}, PlotStyle → Red]]
```



References:

[1] Edward A. Bender, Asymptotic methods in enumeration, SIAM Review 16 (1974), no. 4, 485-515
[2] Kotěšovec V., Asymptotic of implicit functions if Fww = 0, extension of theorem by Bender, website 19.1.2014
[3] OEIS - The On-Line Encyclopedia of Integer Sequences
[4] P. Flajolet and R. Sedgewick, Analytic Combinatorics, 2009, p. 469
[5] Kotěšovec V., Interesting asymptotic formulas for binomial sums, website 9.6.2013
[6] Kotěšovec V., Asymptotic of a sums of powers of binomial coefficients * x^k, website 20.9.2012
[7] Kotěšovec V., Asymptotic of sequences A244820, A244821 and A244822, website (and OEIS) 11.7.2014

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