Interesting asymptotic formulas for binomial sums

(*Václav Kotěšovec, published 9.6.2013, extended 28.6.2013*)

In October and November 2012 I discovered over 400 asymptotic formulas for sequences in the OEIS (On-Line Encyclopedia of Integer Sequences). Most of which are certainly new. This article is a selection of the most interesting. In addition, formulas are more readable if are published in classical mathematical format than in text format only.

Content

	$f(k,n)$ = powers of k or n				
	1	n^k	k^n	k^k	$k^n n^k$
$\sum \binom{n}{k} * f(k,n)$	2^n	$(n+1)^n$	A072034	A086331	A072035

$\sum \binom{n}{k}^k$	A167008
$\sum \binom{n}{k}^n$	A167010
$\sum \binom{kn}{k}$	A226391
$\sum \binom{kn}{n}$	A096131
$\sum \binom{n^2}{kn}$	A167009

$$\sum \binom{n+k}{n} x^k$$

(for more sums of this type see [2] and [3])

1) Basic binomial sums

OEIS - A072034

$$\sum_{k=1}^{n}\binom{n}{k}k^n \sim \frac{\left(\frac{n}{e * LambertW\left(\frac{1}{e}\right)}\right)^n}{\sqrt{1+LambertW\left(\frac{1}{e}\right)}} \sim 0.88441409660870959\ldots * (1.3210997620156174 57\ldots * n)^n$$

We find the maximal term with the help of Stirling's formula. The maximum is a point where the first derivative is equal to zero.

```
stirling[n_] := n^n / E^n * Sqrt[2 * Pi * n];
binom[n_, k_] := stirling[n] / stirling[k] / stirling[n - k];
FullSimplify[D[Simplify[binom[n, r * n] * (r * n) ^n], r]]
```

$$\frac{n^{\frac{3}{2}+n}\,(n\,r)^{-\frac{3}{2}+n-n\,r}\,(n-n\,r)^{-\frac{1}{2}+n\,(-1+r)}\,(1+2\,n\,(-1+r)-2\,r+2\,n\,(-1+r)\,r\,(-\mathrm{Log}[n\,r]+\mathrm{Log}[n-n\,r]))}{2\sqrt{2\,\pi}\,(-1+r)}$$

```
Limit[((1 + 2 n (-1 + r) - 2 r + 2 n (-1 + r) r (-Log[n r] + Log[n - n r]))) / n, n → Infinity]
```

2 (-1 + r) (1 + r Log[1 - r] - r Log[r])

```
Solve[1 + r Log[1 - r] - r Log[r] == 0]
```

$$\left\{\left\{r \to \frac{1}{1+\mathrm{ProductLog}\left[\frac{1}{e}\right]}\right\}\right\}$$

(ProductLog = LambertW)

```
N[1 / (1 + LambertW[1/e]), 20]
```

0.78218829428019990122

r is the root of the equation

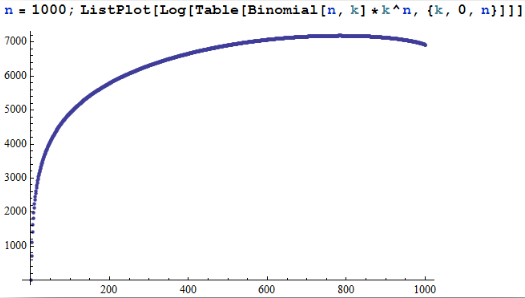
$$\left(\frac{r}{1-r}\right)^r = e$$

```
FindRoot[(r / (1 - r)) ^r == E, {r, 1 / 2}, WorkingPrecision → 50][[1]]
```

r → 0.78218829428019990122029707592674478018190840396630

The maximal term in the sum is at position $r * n$, following graph is in the logarithmical scale.

```
n = 1000; ListPlot[Log[Table[Binomial[n, k] * k^n, {k, 0, n}]]]
```



The value at the maximum is

```
PowerExpand[FullSimplify[binom[n, r*n] * (r*n)^n /. r → 1/(1 + w)]]
```

$$\frac{n^{-\frac{1}{2}+n}\, w^{-\frac{1}{2}-\frac{n w}{1+w}}\, (1+w)}{\sqrt{2\pi}}$$

$$\frac{n^{n-\frac{1}{2}} * (w+1) * w^{-\frac{nw}{w+1}-\frac{1}{2}}}{\sqrt{2\pi}} = \frac{(1+w)}{\sqrt{2\pi n w}} * \left(\frac{n}{ew}\right)^{n}$$

$$r = \frac{1}{1+w} = \frac{1}{1 + LambertW\left(\frac{1}{e}\right)}$$

Now we compute contributions of other terms $k = n * r \pm m$

$$\mathrm{T}(m) = \frac{\binom{n}{nr+m} * (nr+m)^n}{\binom{n}{nr} * (nr)^n}$$

```
lognfak[n_] := n*Log[n] - n + 1/2*Log[n] + 1/2*Log[2*Pi] + 1/(12*n);
logbinom[n_, k_] := lognfak[n] - lognfak[k] - lognfak[n - k];
sLog[k_, n_] := Log[n] + k/n - 1/2*(k/n)^2 + 1/3*(k/n)^3;
Simplify[logbinom[n, r*n + m] + n*Log[r*n + m] -
  (logbinom[n, r*n] + n*Log[r*n])]
```

$$\frac{1}{12}\Bigg(\frac{1}{m+n(-1+r)} + \frac{1}{n r} + \frac{1}{n - n r} - \frac{1}{m + n r} + 6\,\mathrm{Log}[n r] - 12\, n\, \mathrm{Log}[n r] + 12\, n\, r\, \mathrm{Log}[n r] + 6\,\mathrm{Log}[n - n r] + 12\,(n - n r)\,\mathrm{Log}[n - n r] - 6\,\mathrm{Log}[-m + n - n r] + 12\,(m + n(-1+r))\,\mathrm{Log}[-m + n - n r] - 6\,\mathrm{Log}[m + n r] + 12\, n\,\mathrm{Log}[m + n r] - 12\,(m + n r)\,\mathrm{Log}[m + n r]\Bigg)$$

We apply the first three terms from Taylor series (near 0)

$$\log(1+\mathrm{z}) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

and approximate

```
FullSimplify[
 1/12 (1/(m + n (-1 + r)) + 1/(n r) + 1/(n - n r) - 1/(m + n r) + 6 Log[n r] - 12 n Log[n r] +
   12 n r Log[n r] + 6 Log[n - n r] + 12 (n - n r) Log[n - n r] -
   6 sLog[-m, n - n r] + 12 (m + n (-1 + r)) sLog[-m, n - n r] -
   6 sLog[m, n r] + 12 n sLog[m, n r] - 12 (m + n r) sLog[m, n r])]
```

$$\frac{1}{12}\Bigg(\frac{1}{m+n(-1+r)} + \frac{1}{n^3}\Bigg(\frac{2 m^3 (-1+2m)}{(-1+r)^3} + \frac{(3-2m)\, m^2 n}{(-1+r)^2} + \frac{6(-1+m)\, m\, n^2}{-1+r} - \frac{2 m^3 (1+2m-2n)}{r^3} + \frac{m^2 (3+2m-6n)\, n}{r^2} + \frac{(1-6m(1+m-2n))\, n^2}{r}\Bigg) + \frac{1}{n - n r} - \frac{1}{m + n r} - 12\, m\, \mathrm{Log}[n r] + 12\, m\, \mathrm{Log}[n - n r]\Bigg)$$

Last two terms can be simplified

$$12m\log\big(n(1-r)\big) - 12m\log(nr) = 12m\,(\log(1-r) - \log(r)) = \frac{12m}{r}$$

If $m \sim n^{\varepsilon}$, then	if	Log T(m) →	T(m) →
	$\varepsilon < 1/2$	0	1
	$\varepsilon = 1/2$	$\neq 0$	$\neq 0$
	$\varepsilon > 1/2$	$-\infty$	0

```
r = 0.7821882942801999012202970759267447801819084;
Table[
  {eps,
   Limit[
    1/12 (1/(m + n (-1 + r)) + 1/n^3 ((2 m^3 (-1 + 2 m))/(-1 + r)^3 + ((3 - 2 m) m^2 n)/(-1 + r)^2 + (6 (-1 + m) m n^2)/(-1 + r) - (2 m^3 (1 + 2 m - 2 n))/r^3 + (m^2 (3 + 2 m - 6 n) n)/r^2 + ((1 - 6 m (1 + m - 2 n)) n^2)/r) +
      1/(n - n r) - 1/(m + n r) - 12 m / r) /. m -> n^eps, n -> Infinity]}, {eps, 0, 1.5, 0.1}] // MatrixForm
```

0.	0.
0.1	0.
0.2	0.
0.3	0.
0.4	0.
0.5	-3.75203
0.6	-∞
0.7	-∞
0.8	-∞
0.9	-∞
1.	-∞
1.1	-∞
1.2	-∞
1.3	-∞
1.4	-∞
1.5	-∞

0.495	0.
0.496	0.
0.497	0.
0.498	0.
0.499	0.
0.5	-3.75203
0.501	-∞
0.502	-∞
0.503	-∞
0.504	-∞
0.505	-∞

```
Limit[
 1/12
  (1/(m + n (-1 + r)) + 1/n^3
    ((2 m^3 (-1 + 2 m))/(-1 + r)^3 + ((3 - 2 m) m^2 n)/(-1 + r)^2 + (6 (-1 + m) m n^2)/(-1 + r) -
     (2 m^3 (1 + 2 m - 2 n))/r^3 + (m^2 (3 + 2 m - 6 n) n)/r^2 + ((1 - 6 m (1 + m - 2 n)) n^2)/r) +
   1/(n - n r) - 1/(m + n r) - 12 m / r) /. m -> c * Sqrt[n], n -> Infinity]
```

$$\frac{c^2}{2\,(-1+r)\,r^2}$$

$$c^2 = \frac{m^2}{n}$$

$$\sum_{k=1}^{n} \binom{n}{k} k^n \sim \frac{(1+w)}{\sqrt{2\pi n w}} * \left(\frac{n}{ew}\right)^n * \sum_{m=-\infty}^{m=+\infty} e^{-\frac{m^2}{2\,n\,(1-r)\,r^2}}$$

But

$$\sum_{k=-\infty}^{k=+\infty} e^{-\frac{k^2}{N}} \sim \int_{-\infty}^{\infty} e^{-\frac{x^2}{N}}\,dx = \sqrt{\pi N}$$

Here is

$$N = 2n\,(1-r)\,r^2 = \frac{2nw}{(w+1)^3}$$

and the final asymptotic expansion is

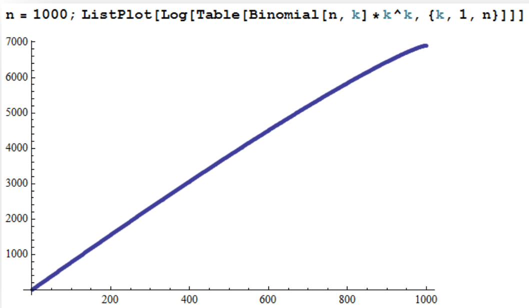
$$\sum_{k=1}^{n} \binom{n}{k} k^n \sim \frac{(1+w)}{\sqrt{2\pi n w}} * \left(\frac{n}{ew}\right)^n * \sqrt{\pi * \frac{2nw}{(w+1)^3}} = \left(\frac{n}{ew}\right)^n * \sqrt{\frac{1}{(w+1)}} = \frac{\left(\frac{n}{e * LambertW\left(\frac{1}{e}\right)}\right)^n}{\sqrt{1 + LambertW\left(\frac{1}{e}\right)}}$$

A086331 (for $k = 0$ is the value = 1)

$$\sum_{k=0}^{n} \binom{n}{k} k^k \sim e^{1/e} * n^n * \left(1 + \frac{1}{2en}\right)$$

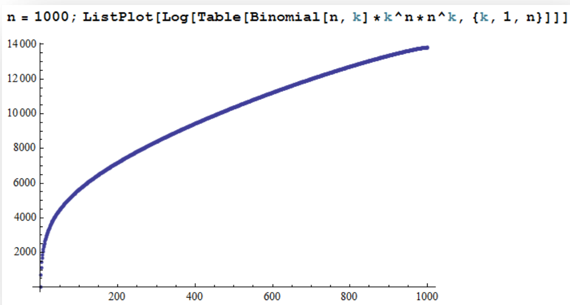
```
stirling[n_] := n^n / E^n * Sqrt[2 * Pi * n];
binom[n_, k_] := stirling[n] / stirling[k] / stirling[n - k];
FullSimplify[D[Simplify[binom[n, r * n] * (r * n) ^ (r * n)], r]]

n^(3/2+n) (n - n r)^(-1/2+n (-1+r)) (1 + 2 (-1 + n (-1 + r)) r + 2 n (-1 + r) r Log[n - n r])
--------------------------------------------------------------------------------------------
                          2 Sqrt[2 π] (-1 + r) (n r)^(3/2)

Limit[(1 + 2 (-1 + n (-1 + r)) r + 2 n (-1 + r) r Log[n - n r]) / (n * Log[n]), n → Infinity]
2 (-1 + r) r
```

$r = 1$ and the maximal term is at position $k = r * n = n$



$$\sum_{k=0}^{n} \binom{n}{k} k^k \sim n^n * \sum_{k=0}^{\infty} \lim_{n\to\infty} \frac{(n-k)^{n-k} \binom{n}{n-k}}{n^n} = n^n * \sum_{k=0}^{\infty} \frac{1}{e^k k!} = e^{1/e}\, n^n$$

A072035 - the maximal term is at position $k = n$



$$\sum_{k=1}^{n} \binom{n}{k} k^n n^k \sim n^{2n} * \sum_{k=0}^{\infty} \lim_{n\to\infty} \frac{n^{n-k} (n-k)^n \binom{n}{n-k}}{n^{2n}} = n^{2n} * \sum_{k=0}^{\infty} \frac{1}{e^k k!} = e^{1/e}\, n^{2n}$$

A167008

$$\lim_{n\to\infty}\left(\sum_{k=0}^{n}\binom{n}{k}^{k}\right)^{\frac{1}{n^2}} = \frac{1}{r^{r^2} * (1-r)^{r*(1-r)}} = r^{\frac{r^2}{1-2r}} = (1-r)^{-\frac{r}{2}} = 1.533628065110458582053143\,...$$

where r = 0.70350607643066243... (A220359) is the root of the equation

$$(1-r)^{2r-1} = r^{2r}$$

(see also A219206)

We find the maximal term with the help of Stirling's approximation

```
stirling[n_] := n^n / E^n * Sqrt[2 * Pi * n];
binom[n_, k_] := stirling[n] / stirling[k] / stirling[n - k];
FullSimplify[D[Simplify[binom[n, k]^k], k]]
```

$$\frac{1}{k-n} 2^{-1-k} \left(k^{-\frac{1}{2}-k} n^{\frac{1}{2}+n} (-k+n)^{-\frac{1}{2}+k-n}\right)^{k} \pi^{-k} \left(-2^{1+\frac{k}{2}} \pi^{k/2} \left(k + (k-n)\left(k\,\mathrm{Log}[k] - k\,\mathrm{Log}[-k+n] - \mathrm{Log}\left[k^{-\frac{1}{2}-k} n^{\frac{1}{2}+n} (-k+n)^{-\frac{1}{2}+k-n}\right]\right)\right) + (2\pi)^{k/2} (n + (-k+n)\,\mathrm{Log}[2\pi])\right)$$

```
FullSimplify[
 PowerExpand[
  (-2^(1 + k/2) π^(k/2) (k + (k - n) (k Log[k] - k Log[-k + n] - Log[k^(-1/2 - k) n^(1/2 + n) (-k + n)^(-1/2 + k - n)])) +
     (2 π)^(k/2) (n + (-k + n) Log[2 π])) /. k → (r * n)]]
```

$$(2\pi)^{\frac{n r}{2}} (n + (n - n r)\,\mathrm{Log}[2\pi]) - 2^{1+\frac{n r}{2}} \pi^{\frac{n r}{2}} \left(n r + \frac{1}{2} n (-1+r) (2 n (-1+2 r)\,\mathrm{Log}[n] + (1+4 n r)\,\mathrm{Log}[r] + (1+2 n-4 n r)\,\mathrm{Log}[n-n r])\right)$$

```
Limit[(n r + 1/2 n (-1 + r) (2 n (-1 + 2 r) Log[n] + (1 + 4 n r) Log[r] + (1 + 2 n - 4 n r) Log[n - n r])) / n^2,
 n → Infinity]
```

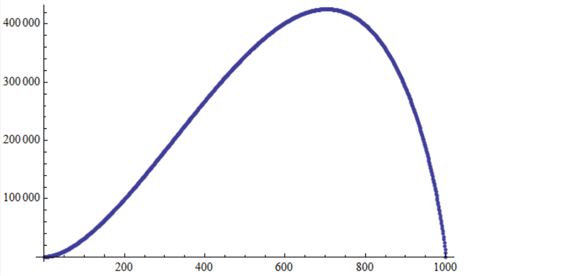
$$-(-1+r)\,((-1+2 r)\,\mathrm{Log}[1-r] - 2 r\,\mathrm{Log}[r])$$

```
FindRoot[(1 - r)^(2 r - 1) == r^(2 r), {r, 1/2}, WorkingPrecision → 50]
{r → 0.70350607643066243096929661621777095213246845742428}
```

The maximal term is asymptotically at position $k = r * n$, where r is the root of the equation

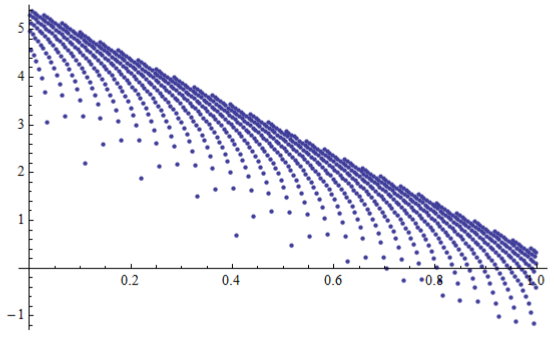
$$(1-r)^{2r-1} = r^{2r}$$

```
n = 1000; ListPlot[Log[Table[Binomial[n, k]^k, {k, 0, n}]]]
```

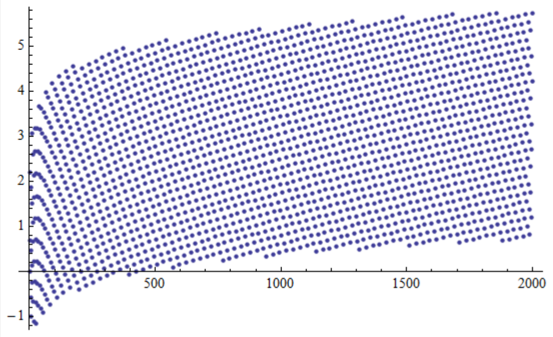


Complication is that $r * n$ is not a integer, see following graphs with distributions of residues and differences.

```
ListPlot[Table[
  kreal = n * 0.70350607643066243;
  k = Floor[n * 0.70350607643066243];
  val0 = N[Log[Binomial[n, k]^k], 10];
  val1 = N[Log[Binomial[n, k + 1]^(k + 1)], 10];
  {kreal - Floor[kreal], val0 - val1}, {n, 1, 1000}]]
```



```
ListPlot[Table[
  kreal = n * 0.70350607643066243;
  k = Floor[n * 0.70350607643066243];
  val0 = N[Log[Binomial[n, k]^k], 10];
  val1 = N[Log[Binomial[n, k + 1]^(k + 1)], 10];
  val0 - val1, {n, 1, 2000}]]
```



For term near maximum and $k = r * n$ is

$$\frac{\binom{n}{k}^k}{\binom{n}{k+1}^{k+1}} = \left(\frac{k+1}{n-k}\right)^{k+1} * \frac{1}{\binom{n}{rn}} = \left(\frac{rn+1}{n-rn}\right)^{rn+1} * \frac{1}{\binom{n}{rn}}$$

$$\left(\frac{rn+1}{n-rn}\right)^{rn+1} \sim \frac{e * r * \left(\frac{r}{1-r}\right)^{nr}}{1-r}$$

From Stirling's approximation

$$\binom{n}{rn} \sim \frac{1}{r^{nr}(1-r)^{n(1-r)} * \sqrt{2\pi nr(1-r)}}$$

Together

$$\left(\frac{rn+1}{n-rn}\right)^{rn+1} * \frac{1}{\binom{n}{rn}} \sim e * r * \sqrt{\frac{2\pi rn}{1-r}} * \left(\frac{r^{2r}}{(1-r)^{2r-1}}\right)^n$$

But for r at the maximum is

$$(1-r)^{2r-1} = r^{2r}$$

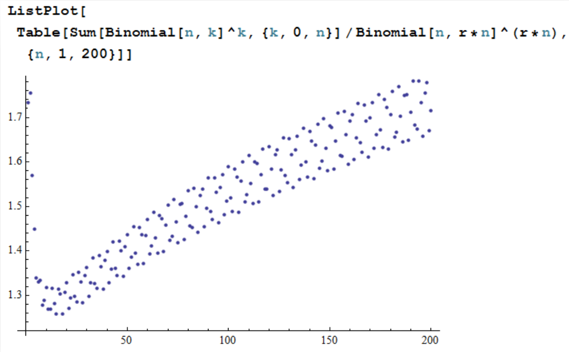
and therefore for $k = r * n$ is

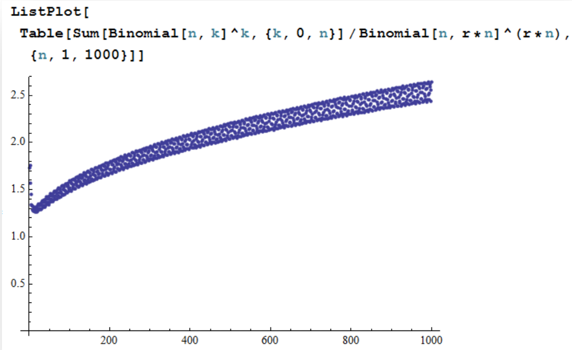
$$\lim_{n\to\infty} \frac{\binom{n}{k}^k}{\binom{n}{k+1}^{k+1}} * \frac{1}{\sqrt{n}} = e * r * \sqrt{\frac{2\pi r}{1-r}} = 7.38377232346663224248764224531926185 \ldots$$

```
Limit[FullSimplify[Binomial[n, k]^k / Binomial[n, k + 1]^(k + 1) /. k → r * n] / Sqrt[n] /.
  FindRoot[(1 - r)^(2 r - 1) == r^(2 r), {r, 1 / 2}, WorkingPrecision → 50][[1]], n → Infinity]
7.3837723234666322424876422453192618506957882140
```

$$\frac{\sum_{k=0}^{n}\binom{n}{k}^{k}}{\binom{n}{rn}^{rn}} < c * n * \frac{1}{\sqrt{n}} = O(\sqrt{n})$$

For this function only lower and upper bound exists, not exact limit or asymptotic.





But following limit exists

$$\lim_{n\to\infty}\left(\sum_{k=0}^{n}\binom{n}{k}^{k}\right)^{\frac{1}{n^2}} = \lim_{n\to\infty}\binom{n}{rn}^{\frac{r}{n}} = (1-r)^{(r-1)r}r^{-r^2} = 1.53362806511045858205314 3\ldots$$

```
FullSimplify[PowerExpand[Simplify[binom[n, n*r]^(r/n)]]]
```

$$n^{-(-1+r)\,r}\,(2\,\pi)^{-\frac{r}{2\,n}}\,r^{-\frac{r\,(1+2\,n\,r)}{2\,n}}\,(n-n\,r)^{\frac{\left(-\frac{1}{2}+n\,(-1+r)\right)\,r}{n}}$$

```
Limit[FullSimplify[PowerExpand[Simplify[binom[n, n*r]^(r/n)]]], n → Infinity]
```

$$(1-r)^{(-1+r)\,r}\,r^{-r^2}$$

```
Limit[FullSimplify[PowerExpand[Simplify[binom[n, n*r]^(r/n)]]], n → Infinity] /.
 FindRoot[(1 - r)^(2 r - 1) == r^(2 r), {r, 1/2}, WorkingPrecision → 50][[1]]
```

```
1.5336280651104585820531430004540738528159363029558
```

A167009

$$\sum_{k=0}^{n} \binom{n^2}{kn} \sim c * \frac{2^{n^2+\frac{1}{2}}}{n\sqrt{\pi}}$$

where

$$c = \sum_{k=-\infty}^{k=+\infty} e^{-2k^2} = 1.271341522189 \ldots$$

if n is even (see A218792)
and

$$c = \sum_{k=-\infty}^{k=+\infty} e^{-2*\left(k+\frac{1}{2}\right)^2} = 1.23528676585389 \ldots$$

if n is odd

Proof: We find the maximal term with the help of Stirling's approximation

```
stirling[n_] := n^n / E^n * Sqrt[2 * Pi * n];
binom[n_, k_] := stirling[n] / stirling[k] / stirling[n - k];
Simplify[D[Simplify[binom[n^2, r * n^2]], r]]
```

$$\frac{1}{2\sqrt{2\pi}} \left(n^2\right)^{\frac{5}{2}+n^2} \left(-n^2(-1+r)\right)^{-\frac{3}{2}+n^2(-1+r)} \left(n^2 r\right)^{-\frac{3}{2}-n^2 r} \left(-1+2r-2n^2(-1+r)\, r\, \mathrm{Log}\left[-n^2(-1+r)\right]+2n^2(-1+r)\, r\, \mathrm{Log}\left[n^2 r\right]\right)$$

```
Limit[(-1 + 2 r - 2 n^2 (-1 + r) r Log[-n^2 (-1 + r)] + 2 n^2 (-1 + r) r Log[n^2 r]) / n^2,
 n → Infinity]
```

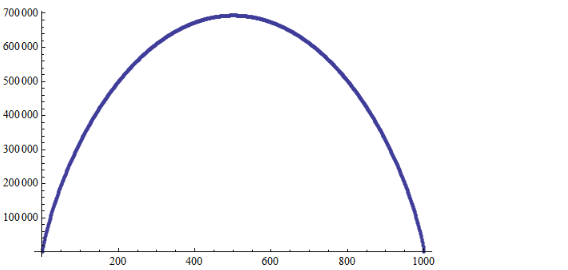
$$-2(-1+r)\, r\, (\mathrm{Log}[1-r]-\mathrm{Log}[r])$$

The maximal term in the sum is at position

$$r = 1/2$$

$$k = n/2$$

```
n = 1000; ListPlot[Log[Table[Binomial[n^2, k * n], {k, 0, n}]]]
```



The value at the maximum is

```
PowerExpand[Simplify[binom[n^2, n^2 / 2]]]
```

$$\frac{2^{\frac{1}{2}+n^2}}{n\sqrt{\pi}}$$

If n is even then

$$\sum_{k=0}^{n}\binom{n^2}{kn} \sim \frac{2^{n^2+\frac{1}{2}}}{\sqrt{\pi n}} * \sum_{k=-\infty}^{k=+\infty} \lim_{n\to\infty} \frac{\binom{n^2}{n\left(\frac{n}{2}+k\right)}}{\binom{n^2}{\frac{n^2}{2}}} = \frac{2^{n^2+\frac{1}{2}}}{\sqrt{\pi n}} * \sum_{k=-\infty}^{k=+\infty} e^{-2k^2}$$

If n is odd then

$$\sum_{k=0}^{n}\binom{n^2}{kn} \sim \frac{2^{n^2+\frac{1}{2}}}{\sqrt{\pi n}} * \sum_{k=-\infty}^{k=+\infty} \lim_{n\to\infty} \frac{\binom{n^2}{n\left(\frac{n}{2}+k+\frac{1}{2}\right)}}{\binom{n^2}{\frac{n^2}{2}}} = \frac{2^{n^2+\frac{1}{2}}}{\sqrt{\pi n}} * \sum_{k=-\infty}^{k=+\infty} e^{-\frac{1}{2}(2k+1)^2}$$

```
Limit[Binomial[n^2, n * (n / 2 + k)] / Binomial[n^2, n^2 / 2], n → Infinity]
```

$e^{-2\,k^2}$

```
Limit[Binomial[n^2, n * (n / 2 + k + 1 / 2)] / Binomial[n^2, n^2 / 2], n → Infinity]
```

$e^{-\frac{1}{2}\,(1+2\,k)^2}$

A167010

$$\sum_{k=0}^{n} \binom{n}{k}^n \sim c * e^{-1/4} * \frac{2^{n^2+\frac{n}{2}}}{(\pi n)^{n/2}}$$

where

$$c = \sum_{k=-\infty}^{k=+\infty} e^{-2k^2} = 1.271341522189 \ldots$$

if n is even (see A218792)
and

$$c = \sum_{k=-\infty}^{k=+\infty} e^{-2*\left(k+\frac{1}{2}\right)^2} = 1.23528676585389 \ldots$$

if n is odd

Proof: We find the maximal term with the help of Stirling's approximation

```
stirling[n_] := n^n / E^n * Sqrt[2 * Pi * n];
binom[n_, k_] := stirling[n] / stirling[k] / stirling[n - k];
Simplify[D[Simplify[binom[n, r * n]^n], r]]
```

$$-\frac{1}{(-1+r)\,r} 2^{-1-\frac{n}{2}}\, n\, \pi^{-n/2} \left(n^{\frac{1}{2}+n} (n\,r)^{-\frac{1}{2}-n\,r} (n-n\,r)^{-\frac{1}{2}+n\,(-1+r)}\right)^n$$
$$(-1+2\,r+2\,n\,(-1+r)\,r\,\mathrm{Log}[n\,r]-2\,n\,(-1+r)\,r\,\mathrm{Log}[n-n\,r])$$

```
Limit[(-1 + 2 r + 2 n (-1 + r) r Log[n r] - 2 n (-1 + r) r Log[n - n r]) / n, n → Infinity]
```

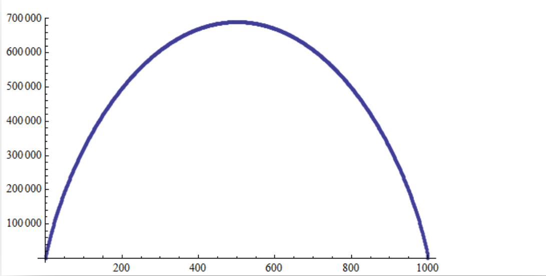
$$-2\,(-1+r)\,r\,(\mathrm{Log}[1-r]-\mathrm{Log}[r])$$

$$r = 1/2$$

The maximal term in the sum is at position

$$k = n/2$$

```
n = 1000; ListPlot[Log[Table[Binomial[n, k]^n, {k, 0, n}]]]
```



From simple form of Stirling's formula we obtain main asymptotic term,

```
PowerExpand[Simplify[binom[n, n / 2]^n]]
```

$$2^{n\left(\frac{1}{2}+n\right)}\, n^{-n/2}\, \pi^{-n/2}$$

but such result is not exact. For more precise asymptotic we must use in this case better approximation:

```
stirlingx[n_] := (n^n / E^n * Sqrt[2 * Pi * n] * (1 + (1 / 12) / n));
binomx[n_, k_] := stirlingx[n] / stirlingx[k] / stirlingx[n - k];
PowerExpand[Simplify[binomx[n, n / 2] ^n]]
```

$$2^{n\left(\frac{1}{2}+n\right)}\, 3^{n}\, n^{n/2}\, (1+6\,n)^{-2\,n}\, (1+12\,n)^{n}\, \pi^{-n/2}$$

$$\text{Limit}\left[\left(2^{n\left(\frac{1}{2}+n\right)}\, 3^{n}\, n^{n/2}\, (1+6\,n)^{-2\,n}\, (1+12\,n)^{n}\, \pi^{-n/2}\right) \Big/ \left(2^{n\left(\frac{1}{2}+n\right)}\, n^{-n/2}\, \pi^{-n/2}\right),\ n \to \text{Infinity}\right]$$

$$\frac{1}{e^{1/4}}$$

$$\binom{n}{n/2}^{n} \sim \frac{1}{e^{1/4}} * 2^{n\left(n+\frac{1}{2}\right)} n^{-\frac{n}{2}} \pi^{-\frac{n}{2}}$$

Same result we obtain also with Maple

$$asympt\left(\left(\text{binomial}\left(n, \frac{n}{2}\right)\right)^{n}, n, 2\right)$$

$$\frac{\left(e^{-\frac{1}{4}} + O\left(\frac{1}{n^{2}}\right)\right) 2^{n^{2}} \sqrt{2^{n}} \sqrt{\left(\frac{1}{n}\right)^{n}}}{\sqrt{\pi^{n}}}$$

Now, if n is even then

$$\sum_{k=0}^{n} \binom{n}{k}^{n} \sim e^{-1/4} * \frac{2^{n^2+\frac{n}{2}}}{(\pi n)^{n/2}} * \sum_{k=-\infty}^{k=+\infty} \lim_{n\to\infty} \frac{\binom{n}{\frac{n}{2}+k}^{n}}{\binom{n}{n/2}^{n}} = e^{-1/4} * \frac{2^{n^2+\frac{n}{2}}}{(\pi n)^{n/2}} * \sum_{k=-\infty}^{k=+\infty} e^{-2k^2}$$

If n is odd then

$$\sum_{k=0}^{n} \binom{n}{k}^{n} \sim e^{-1/4} * \frac{2^{n^2+\frac{n}{2}}}{(\pi n)^{n/2}} * \sum_{k=-\infty}^{k=+\infty} \lim_{n\to\infty} \frac{\binom{n}{\frac{n}{2}+k+\frac{1}{2}}^{n}}{\binom{n}{n/2}^{n}} = e^{-1/4} * \frac{2^{n^2+\frac{n}{2}}}{(\pi n)^{n/2}} * \sum_{k=-\infty}^{k=+\infty} e^{-2*\left(k+\frac{1}{2}\right)^2}$$

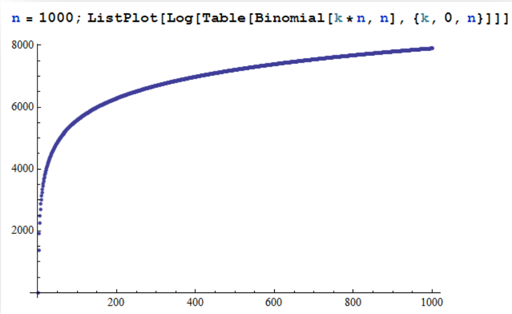
```
Limit[Binomial[n, n / 2 + k] ^n / Binomial[n, n / 2] ^n, n → Infinity]
```

$$e^{-2\,k^2}$$

```
Limit[Binomial[n, n / 2 + k + 1 / 2] ^n / Binomial[n, n / 2] ^n, n → Infinity]
```

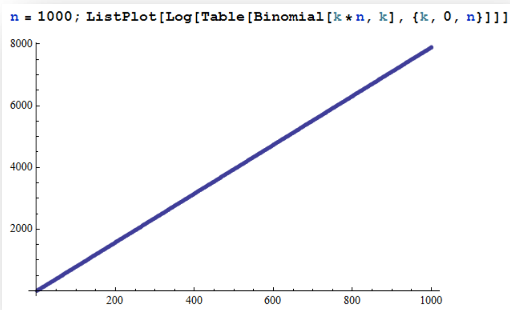
$$e^{-\frac{1}{2}\,(1+2\,k)^2}$$

A096131 - the maximal term in the sum is at position $k = n$



$$\sum_{k=0}^{n}\binom{kn}{n} \sim \binom{n^2}{n} * \sum_{k=0}^{\infty} \lim_{n\to\infty} \frac{\binom{(n-k)*n}{n}}{\binom{n^2}{n}} = \binom{n^2}{n} * \sum_{k=0}^{\infty} e^{-k} = \frac{e}{e-1} * \binom{n^2}{n}$$

A226391



$$\sum_{k=0}^{n}\binom{kn}{k} \sim \binom{n^2}{n} * \sum_{k=0}^{\infty} \lim_{n\to\infty} \frac{\binom{(n-k)*n}{n-k}}{\binom{n^2}{n}} = \binom{n^2}{n}$$

2) Binomial sums with x^k

For $x > \frac{1}{2}$

$$\sum_{k=0}^{n} \binom{n+k}{n} x^k \sim \frac{2x}{2x-1} * \frac{(4x)^n}{\sqrt{\pi n}}$$

For $x = \frac{1}{2}$

$$\sum_{k=0}^{n} \binom{n+k}{n} x^k = \sum_{k=0}^{n} \binom{n+k}{n} \left(\frac{1}{2}\right)^k = 2^n$$

and for $0 < x < \frac{1}{2}$

$$\sum_{k=0}^{n} \binom{n+k}{n} x^k \sim \frac{1}{(1-x)^{n+1}}$$

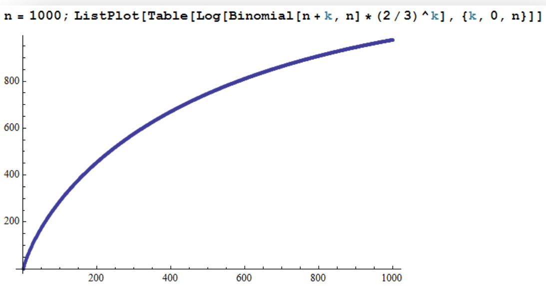
$x = \frac{1}{3}$ A141223, $3^n \sum_{k=0}^{n} \binom{k+n}{n} \frac{1}{3^k}$

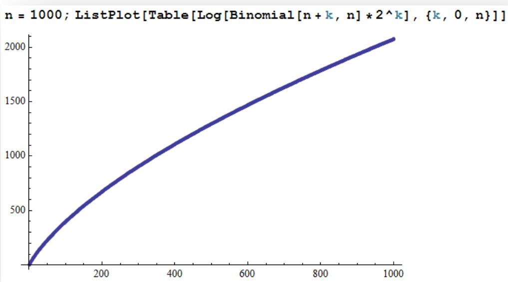
$x = \frac{2}{3}$ A089022, $2^n \binom{2n}{n} - 3^{n-1} \sum_{k=0}^{n-1} \left(\frac{2}{3}\right)^k \binom{n+k}{n} \sim 6 * \frac{8^n}{\sqrt{\pi}\, n^{3/2}}$

$x = 1$ A001700 $= \binom{2n+1}{n+1}$

$x = 2$ A178792

Proof: If $x > \frac{1}{2}$ then the maximal term in the sum is at position $k = n$ (graphs for $x = \frac{2}{3}$ and for $x = 2$)





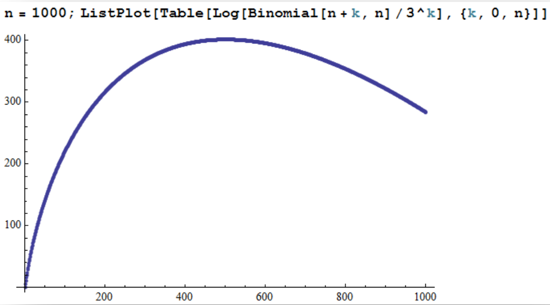
$$\sum_{k=0}^{n} \binom{n+k}{n} x^k \sim \binom{2n}{n} x^n * \sum_{k=0}^{\infty} \lim_{n\to\infty} \frac{\binom{2n-k}{n} x^{n-k}}{\binom{2n}{n} x^n} = \binom{2n}{n} x^n * \sum_{k=0}^{\infty} \frac{1}{(2x)^k} = \frac{2x}{2x-1} * \frac{(4x)^n}{\sqrt{\pi n}}$$

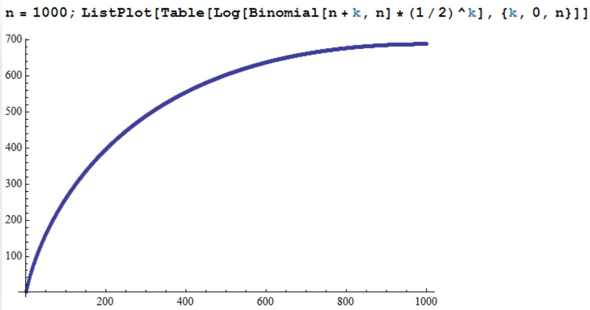
Case $0 < x < \frac{1}{2}$

```
stirling[n_] := n^n / E^n * Sqrt[2 * Pi * n];
binom[n_, k_] := stirling[n] / stirling[k] / stirling[n - k];
FullSimplify[D[Simplify[binom[n + r * n, n] * x^(r * n)], r]]

   1        3/2-n      -3/2-n r              -1/2+n+n r   n r
-------- n        (n r)         (n (1 + r))              x
2 Sqrt[2 π]
 (-1 + 2 n r (1 + r) (-Log[n r] + Log[n (1 + r)] + Log[x]))

Limit[FullSimplify[PowerExpand[(-1 + 2 n r (1 + r) (-Log[n r] + Log[n (1 + r)] + Log[x]))]] /
  n, n → Infinity]

-2 r (1 + r) (Log[r] - Log[1 + r] - Log[x])

Solve[Log[r] - Log[1 + r] - Log[x] == 0, r]

{{r → -x/(-1 + x)}}
```

The maximal term in the sum is at position

$$k = \frac{x}{1-x} * n \qquad \text{(graphs for } x = \tfrac{1}{3} \text{ and for } x = \tfrac{1}{2}\text{)}$$





The value at the maximum is

```
Assuming[{x > 0, x < 1 / 2},
 FullSimplify[PowerExpand[binom[n + r * n, n] * x^ (r * n) /. r -> x/(1 - x)]]]
```

$$\frac{(1-x)^{-n}}{\sqrt{2\pi}\,\sqrt{n\,x}}$$

and contributions of others terms is (with the same method as on page 4)

```
Assuming[{x > 0, x < 1 / 2},
 Limit[binom[n + r * n + m, n] * x^ (r * n + m) / (binom[n + r * n, n] * x^ (r * n)) /.
   r -> x/(1 - x) /. {m -> c * Sqrt[n]}, n -> Infinity]]
```

$$e^{-\frac{c^2 (-1+x)^2}{2x}}$$

where

$$c^2 = \frac{m^2}{n}$$

$$\sum_{k=0}^{n} \binom{n+k}{n} x^k \sim \frac{1}{(1-x)^n * \sqrt{2\pi n x}} * \sum_{m=-\infty}^{m=+\infty} e^{-\frac{m^2(1-x)^2}{2nx}}$$

But

$$\sum_{k=-\infty}^{k=+\infty} e^{-\frac{k^2}{N}} \sim \int_{-\infty}^{\infty} e^{-\frac{x^2}{N}}\,dx = \sqrt{\pi N}$$

Here is

$$N = \frac{2nx}{(1-x)^2}$$

and the final asymptotic expansion for $0 < x < \frac{1}{2}$ is

$$\sum_{k=0}^{n} \binom{n+k}{n} x^k \sim \frac{1}{(1-x)^n * \sqrt{2\pi n x}} * \sqrt{\frac{2\pi n x}{(1-x)^2}} = \frac{1}{(1-x)^{n+1}}$$

Main results from my previous articles (see [2] and [3] for more):

For $p \geq 1,\ x > 0,\ n \to \infty$

$$\sum_{k=0}^{n} \binom{n}{k}^{p} x^{k} \sim \frac{\left(1 + x^{\frac{1}{p}}\right)^{pn+p-1}}{\sqrt{(2\pi n)^{p-1} * p * x^{1-\frac{1}{p}}}}$$

For $p > 0, q \geq 0, n \to \infty$ *is*

$$\sum_{k=0}^{n} \binom{n}{k}^{p} \binom{n+k}{k}^{q} \sim \frac{(1+r)^{qn}}{(1-r)^{pn+p}} * \sqrt{\frac{r\,(1-r^{2})}{(p+q+(p-q)\,r) * (2\pi n)^{p+q-1}}}$$

where r is positive real root of the equation

$$(1-r)^{p} * (1+r)^{q} = r^{p+q}$$

Especially for $p = q > 0$

$$\sum_{k=0}^{n} \binom{n}{k}^{p} \binom{n+k}{k}^{p} \sim \frac{\left(1+\sqrt{2}\right)^{p\,(2n+1)}}{2^{p/2+3/4} * (\pi n)^{p-1/2} * \sqrt{p}}$$

and for $p = 2q > 0$

$$\sum_{k=0}^{n} \binom{n}{k}^{2q} \binom{n+k}{k}^{q} \sim \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{q\,(5n+4)\,-\,3/2}}{5^{1/4} * \sqrt{q\,(2\pi n)^{3q-1}}}$$

For $p = 0,\ q > 0, n \to \infty$ *is*

$$\sum_{k=0}^{n} \binom{n+k}{k}^{q} \sim \frac{2^{(2n+1)*q}}{(2^{q}-1) * (\pi n)^{q/2}}$$

3) Sums with Fibonacci and Lucas numbers

A135961 - sum with Fibonacci numbers

$$\sum_{k=0}^{n} (F_k)^{n-k} \sim c * \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{n^2}{4}} * 5^{-\frac{n}{4}} \sim c * \left(\frac{F_n}{\sqrt{5}}\right)^{\frac{n}{4}}$$

where

$$c = \sum_{k=-\infty}^{k=+\infty} 5^{k/2} * \left(\frac{1+\sqrt{5}}{2}\right)^{-k^2} = 3.5769727481316948565395\ldots$$

(A219781) if n is even
and

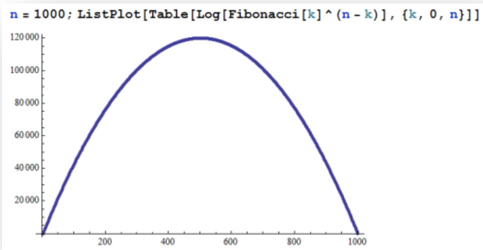
$$c = \sum_{k=-\infty}^{k=+\infty} 5^{\frac{k+\frac{1}{2}}{2}} * \left(\frac{1+\sqrt{5}}{2}\right)^{-\left(k+\frac{1}{2}\right)^2} = 3.5769727390073366345992\ldots$$

if n is odd

Interesting is that first 7 decimal places of both constants are same, but constants are different!

Proof:

$$\sum_{k=0}^{n} (F_k)^{n-k} \sim \sum_{k=0}^{n} 5^{\frac{k-n}{2}} \left(\left(\frac{1}{2}(1+\sqrt{5})\right)^{k}\right)^{n-k}$$



We find the maximal term

```
Simplify[D[(1 / Sqrt[5] * ((1 + Sqrt[5]) / 2) ^ (r * n)) ^ (n - r * n), r]]
```

$$-\frac{1}{2}\, 5^{\frac{1}{2} n (-1+r)} \left(\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^{n r}\right)^{n-n r} n \left(-\mathrm{Log}[5] + 2\,\mathrm{Log}\left[\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^{n r}\right] + 2 n (-1+r)\,\mathrm{Log}\left[\frac{1}{2}\left(1+\sqrt{5}\right)\right]\right)$$

$$\mathrm{Simplify}\left[\mathrm{PowerExpand}\left[\left(2\,\mathrm{Log}\left[\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^{n r}\right] + 2 n (-1+r)\,\mathrm{Log}\left[\frac{1}{2}\left(1+\sqrt{5}\right)\right]\right) \Big/ n\right]\right]$$

$$-2\,(-1+2 r)\,\mathrm{Log}\left[\frac{2}{1+\sqrt{5}}\right]$$

$$r = 1/2$$

The value at the maximum is

```
PowerExpand[Simplify[(1 / Sqrt[5] * ((1 + Sqrt[5]) / 2) ^ (r * n)) ^ (n - r * n) /. r -> 1 / 2]]
```

$$5^{-\frac{n}{4}} \left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^{\frac{n^2}{4}}$$

Now, if n is even then

$$\sum_{k=0}^{n}(F_k)^{n-k} \sim 5^{-\frac{n}{4}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{n^2}{4}} * \sum_{k=-\infty}^{\infty}\frac{\left(F_{n/2+k}\right)^{n/2-k}}{\left(F_{n/2}\right)^{n/2}} \sim 5^{-\frac{n}{4}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{n^2}{4}} * \sum_{k=-\infty}^{\infty} 5^{k/2}\left(\frac{2}{1+\sqrt{5}}\right)^{k^2}$$

if n is odd then

$$\sum_{k=0}^{n}(F_k)^{n-k} \sim 5^{-\frac{n}{4}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{n^2}{4}} * \sum_{k=-\infty}^{\infty}\frac{\left(F_{n/2+k+1/2}\right)^{n/2-k-1/2}}{\left(F_{n/2}\right)^{n/2}} \sim 5^{-\frac{n}{4}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{n^2}{4}} * \sum_{k=-\infty}^{\infty} 5^{k/2+1/4}\left(\frac{2}{1+\sqrt{5}}\right)^{\frac{1}{4}(2k+1)^2}$$

A187780 - similar result for Lucas numbers

$$\sum_{k=0}^{n}(L_k)^{n-k} \sim c * \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{n^2}{4}} \sim c * (L_n)^{n/4}$$

where

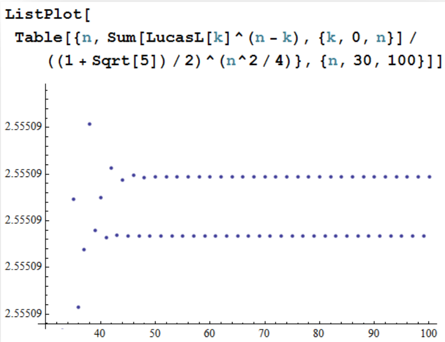
$$c = \sum_{k=-\infty}^{k=+\infty}\left(\frac{1+\sqrt{5}}{2}\right)^{-k^2} = 2.55509346944451877723 0568 \ldots$$

if n is even

and

$$c = \sum_{k=-\infty}^{k=+\infty}\left(\frac{1+\sqrt{5}}{2}\right)^{-\left(k+\frac{1}{2}\right)^2} = 2.555093456793304790966994 \ldots$$

if n is odd



4) Miscellaneous binomial sums

A219614 - sum with Stirling numbers of the second kind

$$a_n = \sum_{k=0}^{n} \binom{n-k+1}{k} k! * S_2(n,k)$$

$$\lim_{n\to\infty} \left(\frac{a_n}{n!}\right)^{\frac{1}{n}} = \frac{3r^2 - 3r + 1}{1 - 2r} = 1.5344563093166842150623 6\ldots$$

where r = 0.410751485627... is the root of the equation

$$(1-2r)^2 + r * (1 - 3r + 3r^2) * LambertW\left(-\frac{e^{-1/r}}{r}\right) = 0$$

For Stirling number first and second kind (in central region!) I use following approximations (in Mathematica notation):

```
S1asy[n_,k_]:=n!/k!*(-Log[-k/n/LambertW[-1,-k/n*Exp[-k/n]]])^k
/(1+k/(n*LambertW[-1,-k/n*Exp[-k/n]]))^n*Sqrt[-k/(2*Pi*n^2*(LambertW[-1,-k/n
*Exp[-k/n]]+1))];

S2asy[n_,k_]:=n!/k! * (n/k+LambertW[-n/k*Exp[-n/k]])^(k-n) / ((-LambertW[-n/k*
Exp[-n/k]])^k * Sqrt[2*Pi*n*(1+LambertW[-n/k*Exp[-n/k]])]);
```

```
FullSimplify[D[binom[n - r*n + 1, r*n]*S2asy[n, r*n]*(r*n)!, r]]
```

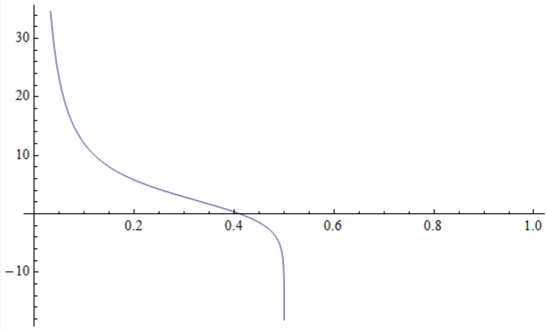
$$\Bigg(n\,(n\,r)^{-\frac{1}{2}-n\,r}\,(1+n-2\,n\,r)^{-\frac{3}{2}+n\,(-1+2\,r)}\,(1+n-n\,r)^{\frac{3}{2}+n-n\,r}\,n!\,\left(-\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)^{-n\,r}\left(\frac{1}{r}+\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)^{n\,(-1+r)}$$
$$\Bigg(n\left(\frac{1}{-1+n\,(-1+r)}+\frac{2}{1+n-2\,n\,r}\right)-2\,n\left(\text{Log}[n\,r]-2\,\text{Log}[1+n-2\,n\,r]+\text{Log}[1+n-n\,r]+\text{Log}\left[-\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right]-\text{Log}\left[\frac{1}{r}+\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right]\right)+$$
$$\left(-\frac{1}{r}+n\left(\frac{1}{-1+n\,(-1+r)}+\frac{2}{1+n-2\,n\,r}\right)-2\,n\left(\text{Log}[n\,r]-2\,\text{Log}[1+n-2\,n\,r]+\text{Log}[1+n-n\,r]+\text{Log}\left[-\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right]-\right.\right.$$
$$\left.\left.\text{Log}\left[\frac{1}{r}+\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right]\right)\right)\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]+\frac{-1+\frac{1-r}{1+\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]}}{r^2}\Bigg)\Bigg)\Bigg/\left(4\,\pi\left(n\left(1+\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)\right)^{3/2}\right)$$

$$\text{Limit}\Bigg[$$
$$\Bigg(n\left(\frac{1}{-1+n\,(-1+r)}+\frac{2}{1+n-2\,n\,r}\right)-2\,n\left(\text{Log}[n\,r]-2\,\text{Log}[1+n-2\,n\,r]+\text{Log}[1+n-n\,r]+\text{Log}\left[-\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right]-\text{Log}\left[\frac{1}{r}+\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right]\right)+$$
$$\left(-\frac{1}{r}+n\left(\frac{1}{-1+n\,(-1+r)}+\frac{2}{1+n-2\,n\,r}\right)-\right.$$
$$\left.2\,n\left(\text{Log}[n\,r]-2\,\text{Log}[1+n-2\,n\,r]+\text{Log}[1+n-n\,r]+\text{Log}\left[-\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right]-\text{Log}\left[\frac{1}{r}+\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right]\right)\right)\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]+$$
$$\frac{-1+\frac{1-r}{1+\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]}}{r^2}\Bigg)\Bigg/n,\ n\to\text{Infinity}\Bigg]$$

$$2\left(2\,\text{Log}[1-2\,r]-\text{Log}[1-r]-\text{Log}[r]-\text{Log}\left[-\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right]+\text{Log}\left[\frac{1}{r}+\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right]\right)\left(1+\text{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)$$

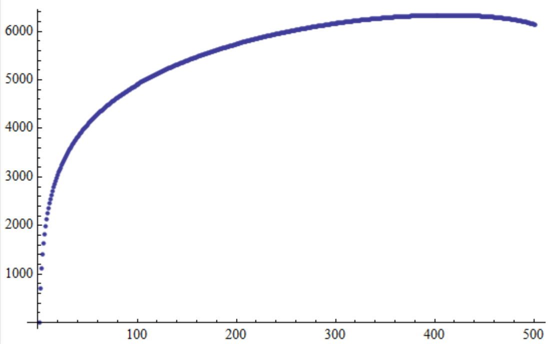
```
FindRoot[2 Log[1 - 2 r] - Log[1 - r] - Log[r] - Log[-ProductLog[-E^(-1/r)/r]] + Log[1/r + ProductLog[-E^(-1/r)/r]] == 0,
  {r, 0.3}, WorkingPrecision → 50]
{r → 0.41075148562708624194639923018891139764505759339773}
```

```
Plot[2 Log[1 - 2 r] - Log[1 - r] - Log[r] - Log[-ProductLog[-E^(-1/r)/r]] +
  Log[1/r + ProductLog[-E^(-1/r)/r]], {r, 0, 1}]
```



Point of the maximum

```
n = 1000;
ListPlot[ParallelTable[Log[Binomial[n - k + 1, k] * StirlingS2[n, k] k!], {k, 0, n}]]
```



(terms for $k > n/2$ are equal to zero)

$$\mathrm{Limit}\left[\left(\left(\left(\left(e^{-1}\, n^{\frac{1}{2n}+1}\, (n\, r)^{-\frac{1}{2n}-r}\, (1+n-2\, n\, r)^{-\frac{3}{2n}+(-1+2\, r)}\, (1+n-n\, r)^{\frac{3}{2n}+1-r}\left(-\mathrm{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)^{-r}\left(\frac{1}{r}+\mathrm{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)^{(-1+r)}\right)\right)\right)\Big/\ (n/E)\right),\ n\to \mathrm{Infinity}\right]$$

$$\left((1-2\, r)^{-1+2\, r}\, (-(-1+r)\, r)^{1-r}\left(-\mathrm{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)^{-r}\left(\frac{1}{r}+\mathrm{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)^{r}\right)\Big/\left(1+r\, \mathrm{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)$$

$$N\left[\left((1-2\, r)^{-1+2\, r}\, (-(-1+r)\, r)^{1-r}\left(-\mathrm{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)^{-r}\left(\frac{1}{r}+\mathrm{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)^{r}\right)\Big/\left(1+r\, \mathrm{ProductLog}\left[-\frac{e^{-1/r}}{r}\right]\right)\ /.\ r \to 0.41075148562708624194639923,\ 20\right]$$

```
1.5344563093166842151
```

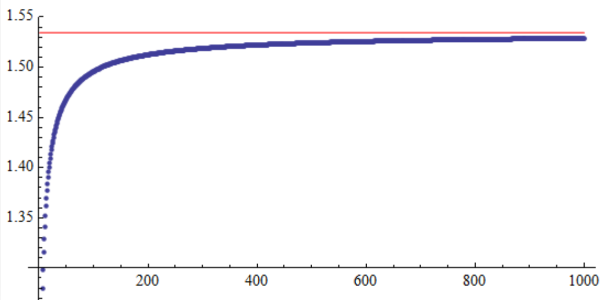
$$\mathrm{PowerExpand}\left[\mathrm{FullSimplify}\left[\left((1-2\, r)^{-1+2\, r}\, (-(-1+r)\, r)^{1-r}\, (-((1-2*r)\verb|^|2/(-r*(1-3*r+3*r\verb|^|2))))^{-r}\left(\frac{1}{r}+(1-2*r)\verb|^|2/(-r*(1-3*r+3*r\verb|^|2))\right)^{r}\right)\Big/\ (1+r*((1-2*r)\verb|^|2/(-r*(1-3*r+3*r\verb|^|2))))\right]\right]$$

$$(1-2\, r)^{-1+2\, r}\, r^{-r}\, (1+3\, (-1+r)\, r)^{1-r}\left(\frac{1}{r}+\frac{-1+r}{1+3\, (-1+r)\, r}\right)^{-r}$$

```
(3 * r^2 - 3 * r + 1) / (1 - 2 * r) /.
 FindRoot[(1 - 2 * r)^2 + r * (1 - 3 * r + 3 * r^2) * LambertW[-E^(-1 / r) / r] == 0,
  {r, 1 / 2}, WorkingPrecision → 50]
1.5344563093166842150623600010206933066951350119 57
```

Numerical verify:

```
Show[
 ListPlot[
  ParallelTable[(Sum[Binomial[n - k + 1, k] * StirlingS2[n, k] k!, {k, 0, n}] / n!)^(1 / n),
   {n, 1, 1000}], PlotRange → {1.3, 1.55}],
 Plot[(3 * r^2 - 3 * r + 1) / (1 - 2 * r) /.
   FindRoot[(1 - 2 * r)^2 + r * (1 - 3 * r + 3 * r^2) * LambertW[-E^(-1 / r) / r] == 0,
    {r, 1 / 2}, WorkingPrecision → 100], {n, 1, 1000}, PlotStyle → Red],
 PlotRange → {1.2, 1.55}]
```



A102743

$$n!\sum_{k=1}^{n+1}\frac{k^{k-1}}{k!} \sim \frac{e^2}{e-1} * n^{n-1}$$

The maximal term is at position $k = n + 1$

$$n!\sum_{k=1}^{n+1}\frac{k^{k-1}}{k!} \sim n^{n-1} * \sum_{k=0}^{\infty}\lim_{n\to\infty}\frac{n!\,(n+1-k)^{n-k}}{n^{n-1} * (n+1-k)!} = n^{n-1} * \sum_{k=0}^{\infty} e^{1-k} = \frac{e^2}{e-1} * n^{n-1}$$

A220452

$$\sum_{k=1}^{n}(2k-3)!!\binom{n}{k} \sim (2n-3)!! * \sqrt{e}$$

Proof: the maximal term is at position $k = n$

$$\sum_{k=1}^{n}(2k-3)!!\binom{n}{k} \sim (2n-3)!! * \frac{\sum_{j=0}^{\infty}(2n-2j-3)!!\binom{n}{n-j}}{(2n-3)!!} \sim (2n-3)!! * \sum_{j=0}^{\infty}\lim_{n\to\infty}\frac{2^{-j}(1-2n)\sqrt{n}}{j!\,(2j-2n+1)\sqrt{n-j}}$$

$$\sim (2n-3)!! * \sum_{j=0}^{\infty}\frac{2^{-j}}{j!} \sim (2n-3)!! * \sqrt{e} \sim n^{n-1} * 2^{n-\frac{1}{2}} * e^{\frac{1}{2}-n}$$

(according with Mathematica, (-1)!!=1)

New asymptotic formulas (extended 28.6.2013)

$$\sum_{k=0}^{\left\lfloor \frac{n}{p} \right\rfloor} \frac{\binom{(p+1)k}{k}\binom{n}{pk}}{pk+1} \sim \frac{\left((p+1)^{\frac{1}{p}+1}+p\right)^{n+\frac{3}{2}}}{\sqrt{2\pi}\, n^{3/2}\, (p+1)^{\frac{3}{2p}+1} p^{n+1}}$$

A007317(n+1) (p=1), A049130(n+1) (p=2), A226974 (p=3), A227035 (p=4), A226910 (p=5)

A135753

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{3^k-1}{2}\right)^{n-k} \sim c * \frac{3^{\frac{n^2}{4}}\, 2^{\frac{n+1}{2}}}{\sqrt{\pi n}}$$

where

$$c = \sum_{k=-\infty}^{\infty} 2^k\, 3^{-k^2} = 1.8862156350800186 \ldots$$

if n is even and

$$c = \sum_{k=-\infty}^{\infty} 2^{k+\frac{1}{2}}\, 3^{-\left(k+\frac{1}{2}\right)^2} = 1.8865940733664341 \ldots$$

if n is odd

A135754

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{4^k-1}{3}\right)^{n-k} \sim c * \frac{2^{\frac{n^2}{2}+n+\frac{1}{2}}}{3^{n/2}\sqrt{\pi n}}$$

where

$$c = \sum_{k=-\infty}^{\infty} 3^k\, 4^{-k^2} = 1.86902676808473931 \ldots$$

if n is even and

$$c = \sum_{k=-\infty}^{\infty} 3^{k+\frac{1}{2}}\, 4^{-\left(k+\frac{1}{2}\right)^2} = 1.87384213421283135 \ldots$$

if n is odd

A135079

$$\sum_{k=0}^{n} \binom{n}{k}\, 3^{k(n-k)} \sim c * \frac{3^{\frac{n^2}{4}}\, 2^{n+\frac{1}{2}}}{\sqrt{\pi n}}$$

where

$$c = \sum_{k=-\infty}^{\infty} 3^{-k^2} = 1.6914596816817 \ldots$$

if n is even and

$$c = \sum_{k=-\infty}^{\infty} 3^{-\left(k+\frac{1}{2}\right)^2} = 1.69061120307521 \ldots$$

if n is odd

A048163, S_2 = Stirling numbers of the second kind

$$a_n = \sum_{k=1}^{n} ((k-1)!)^2 * (S_2(n,k))^2$$

$$\lim_{n\to\infty} \left(\frac{a_n}{n!}\right)^{\frac{1}{n}} = \frac{1}{e \log^2(2)} = 0.7656928576 \ldots$$

A122399

$$a_n = \sum_{k=0}^{n} k!\, k^n * S_2(n,k)$$

$$\lim_{n\to\infty} \left(\frac{a_n}{n!}\right)^{\frac{1}{n}} = \frac{(e^{1/r}+1)\, r^2}{e} = 1.16289952747740081 8845 \ldots$$

where $r = 0.87370243323966833...$ is the root of the equation

$$\frac{1}{e^{-1/r}+1} = -r \operatorname{LambertW}\left(-\frac{e^{-1/r}}{r}\right)$$

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