Superqueens

E 3162 [1986, 566]. *Proposed by Paul Monsky, Brandeis University.*

A superqueen is a piece that moves on a square board like an ordinary chess queen but is permitted to continue along the extended diagonals (one may think of the board as a torus with opposite sides next to each other.) A result of Pólya's that has been rediscovered by others from time to time (see, for example, E 2968 [1979, 309]) is that N superqueens may be placed on an N by N board with no two attacking one another if and only if N is prime to 6.

(a) Is it possible, for each value of N, to place $N-2$ superqueens on an N by N board with no two attacking one another?

(b) For what values of N can $N-1$ superqueens be so positioned on an N by N board?

Solution by the proposer. (a) Such placements are always possible; for each value of N we describe one explicitly. If the ith superqueen is in column x_i and row y_i we shall write q_i for the vector (x_i, y_i) viewed as an element of $Z/N \times Z/N$. It suffices to specify the $q_{i+1} - q_i$ for $1 \leqslant i \leqslant N-3$. The "generic" $q_{i+1} - q_i$ will be $(1,2)$ for N odd and $(1,3)$ for N even; however, there will be a few exceptional values of i for which $q_{i+1} - q_i$ takes another value. For conciseness, we denote the vectors $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(2,3)$, $(2,4)$, $(2,5)$, and $(2,1)$ by A, B, C, D, E, F, G, H and I, respectively. A notation such as $ABBC$ (or more briefly AB^2C) will mean a placement of five superqueens with $q_{i+1} - q_i = A, B, B, C$ for $i = 1, 2, 3, 4$, respectively.

If N is not a multiple of 3 or 4, the solution is easy to describe and in fact more than $N-2$ superqueens are possible (cases 1 and 2 below). The placement of the $N-2$ superqueens is far more elaborate in the remaining cases. The constructions are as follows.

(1) If N is prime to 6, then A^{N-1} places N pairwise non-attacking superqueens.

(2) If $N \equiv \pm 2 \bmod 12$, let $N = 2M$ with $M > 1$. Then $B^{M-1}EB^{M-2}$ places $N-1$ pairwise non-attacking superqueens.

(3) If $N \equiv \pm 3 \bmod 12$, let $N = 6M + 3$ with $M > 0$. Then $A^{M-1}BA^{2M}FA^{M-1}FA^{2M-1}$ suffices.

(4) If $N = 12M + 4$, then A suffices for $M = 0$ and $B^{2M-1}DB^{3M-1}GB^{M}CB^{3M-1}DB^{3M}$ suffices for $M > 0$.

(5) If $N = 12M + 6$, then BAB suffices for $M = 0$ and $B^{2M-1}CB^{4M+1}AFB^{2M-1}HB^{4M}$ suffices for $M > 0$.

(6) If $N = 12M + 8$, then $B^{2M}GB^{3M}DB^{M}AB^{3M}GB^{3M+1}$ suffices.

(7) If $N = 12M$, then A^3BIBA^3 suffices for $M = 1$ and $B^{3M-2}EB^{M-2}HB^{2M-1}AB^{3M-1}FB^{M-1}CB^{2M-1}$ suffices for $M > 1$.

Setting $z_i = x_i + y_i$ and $t_i = x_i - y_i$, it is straightforward but tedious to show that in each placement above the x_i, y_i, z_i, t_i each represent distinct residue classes mod N. Thus no pair of placed superqueens attack each other.

(b) Placing $N-1$ superqueens is possible if and only if N is not divisible by 3 or 4. (1) and (2) of part (a) show the condition is sufficient. To prove necessity, suppose $N - 1$ pairwise nonattacking superqueens have been placed. By translation, we may assume that row N and column N are empty.

Suppose the ith superqueen is in row x_i and column y_i, and set $z_i = x_i + y_i$ and $t_i = x_i - y_i$. Viewing the summations in terms of x_i and y_i, we have $\Sigma z_i = N(N-1)$, $\Sigma t_i = 0$, and $\Sigma(z_i^2 + t_i^2) = 2N(N-1)(2N-1)/3$. Since the superqueens are nonattacking, $\{z_i\}$ and $\{t_i\}$ each occupy $N - 1$ congruence classes mod N; let u, v with $0 \leqslant u, v < N$ be the omitted class for $\{z_i\}, \{t_i\}$, respectively. Then $u + \Sigma z_i \equiv v + \Sigma t_i \equiv \Sigma_{i=0}^{N-1} i \equiv N(N-1)/2 \bmod N$. Therefore, $u \equiv v \equiv N(N-1)/2 \bmod N$. This implies $u = v = 0$ for odd N and $u = v = N/2$ for even N.

When N is odd, we conclude that the z_i's belong to distinct nonzero congruence classes mod N; similarly for $\{t_i\}$. Thus $\Sigma z_i^2 \equiv \Sigma t_i^2 \equiv \Sigma j^2 \bmod N$, so $\Sigma(z_i^2 + t_i^2) \equiv N(N-1)(2N-1)/3 \bmod N$. Since the sum equals $2N(N-1)(2N-1)/3$, we conclude that $(N-1)(2N-1)/3$ must be an integer, which implies that N is not divisible by 3 (or 4, being odd).

When N is even, let $z_0 = t_0 = N/2$. Then the z_i's belong to distinct congruence classes mod N, and similarly for the t_i's. Since N is even, $z_i \equiv j \bmod N$ implies $z_i^2 \equiv j^2 \bmod 2N$. Summing, we find modulo $2N$ that $\Sigma z_i^2 \equiv \Sigma t_i^2 \equiv \Sigma j^2 = N(N-1)(2N-1)/6$. Deleting the terms for $z_0 = t_0 = N/2$, we have $\Sigma_{i=1}^{N-1}(z_i^2 + t_i^2) \equiv N(N-1)(2N-1)/3 - N^2/2 \bmod 2N$. Again the sum equals $2N(N-1)(2N-1)/3$, and we conclude that $N^2/3 - N/4 + 1/6$ is an integer. Again this implies that N cannot be a multiple of 3 or 4.

No other solutions were received.